Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-assumed

ATAR course examination 2019

Marking key

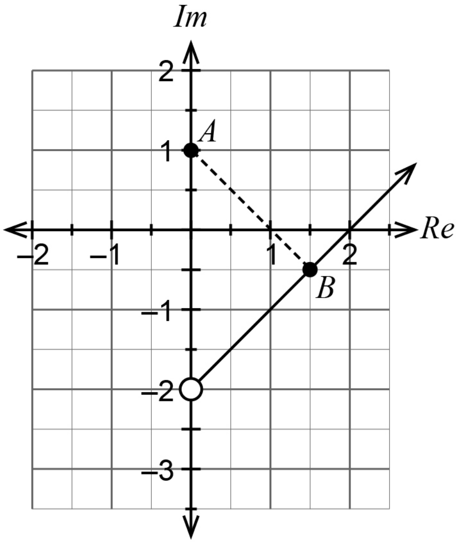
Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2019/63009

Section Two: Calculator-assumed **65% (85 Marks)**

Question 10 **(5 marks)**

The sketch of the locus of a complex number $z = x + iy$ is shown below.



(a) Given that the equation for the above locus is written as $Arg(z - z_0) = k\pi$, determine the value of the constants z_0 and k. (2 marks)

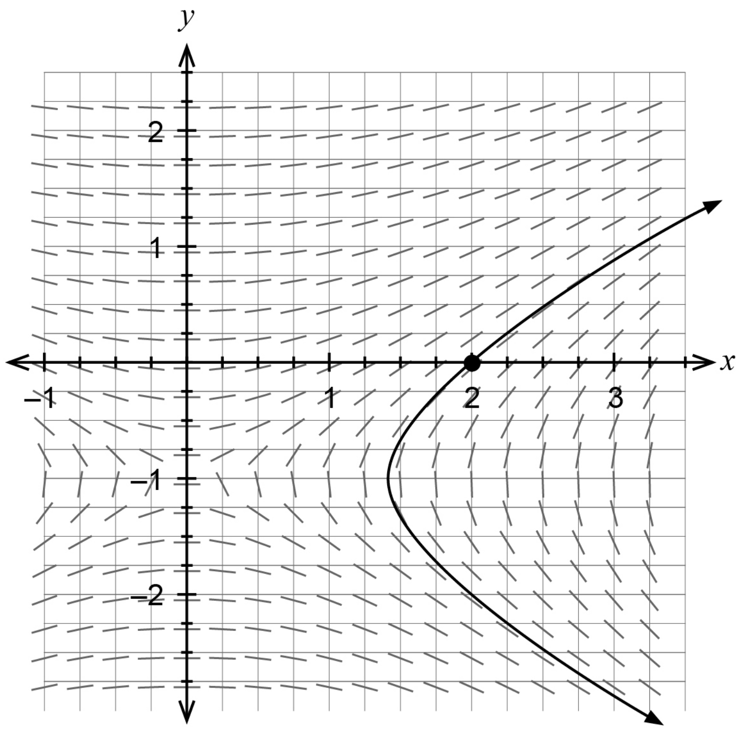
Solution
The equation can be read as the argument of z from z_0 is equal to $k\pi$.   i.e. $Arg(z-(-2i)) = \dfrac{\pi}{4}$ i.e. $z_0 = -2i,\ k = \dfrac{1}{4}$
Specific behaviours
✓ states the correct value for z_0 ✓ states the correct value for k

(b) Determine the minimum value for $\lvert z - i \rvert$ as an exact value. (3 marks)

Solution
We require the minimum distance of a point in the locus from $z = i$ (point A).   This will be the perpendicular distance AB to the locus.   Point B will be the point $(1.5, -0.5i)$. Hence $AB = \sqrt{\left(\dfrac{3}{2}\right)^2 + \left(\dfrac{3}{2}\right)^2} = \sqrt{\dfrac{18}{4}} = \dfrac{3\sqrt{2}}{2}$ Hence the minimum value for $\lvert z - i \rvert = \dfrac{3\sqrt{2}}{2}$.
Specific behaviours
✓ indicates how the minimum value $\lvert z - i \rvert$ is found   ✓ determines coordinates for point B correctly   ✓ determines the minimum value $\lvert z - i \rvert$ correctly

Question 11 **(6 marks)**

The slope field given by $\frac{dy}{dx} = \frac{x}{2y+2}$ is shown in the diagram below.



(a) Calculate the value of the slope field at the point $(2,0)$. (1 mark)

Solution
Evaluating $\frac{dy}{dx}$ when $x=2,\ y=0$: $\frac{dy}{dx} = \frac{2}{2(0)+2} = 1$ i.e. the slope field at $(2,0)$ has a value of 1.
Specific behaviours
✓ evaluates $\frac{dy}{dx}$ correctly

(b) On the diagram above, draw the solution curve that contains the point $(2,0)$. (2 marks)

Solution
Shown above on the diagram.
Specific behaviours
✓ draws a curve that follows the slope field ✓ draws a curve that is vertical at $y=-1$ or symmetric about $y=-1$

Question 11 (continued)

(c) Determine the equation for the solution curve that contains the point $(2,0)$. (3 marks)

Solution
Separating variables obtains $\int(2y+2)\,dy = \int x\,dx$ $\therefore\ y^2+2y = \dfrac{x^2}{2} + c$ Using $(2,0)$: $0^2+2(0) = \dfrac{2^2}{2} + c \quad \therefore\ c=-2$ Equation of the solution through $(2,0)$ is $y^2+2y = \dfrac{x^2}{2} - 2$ Alternatively : $(y+1)^2 = \dfrac{x^2}{2} - 1$ OR $y = \pm\sqrt{\dfrac{x^2}{2} - 1} - 1$
Specific behaviours
✓ separates the variables correctly   ✓ anti-differentiates correctly using a constant   ✓ determines the anti-derivative constant correctly

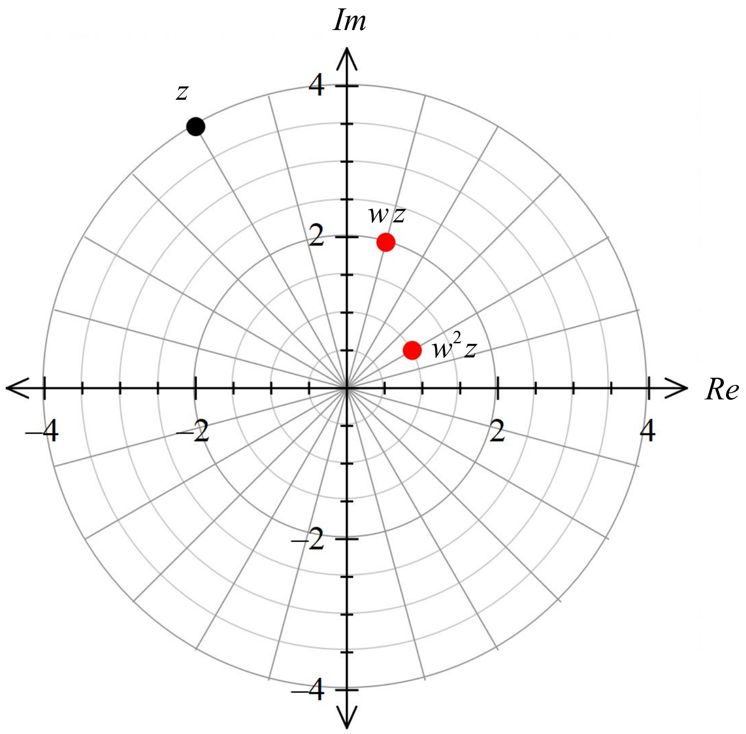
Question 12 **(10 marks)**

Let $w = \dfrac{1-i}{2\sqrt{2}}$.

(a) Express w in the form $w = r\,cis\theta$, where $-\pi < \theta \le \pi$. (2 marks)

Solution
$w = \dfrac{1-i}{2\sqrt{2}} = \dfrac{\sqrt{2}cis\left(-\dfrac{\pi}{4}\right)}{2\sqrt{2}} = \dfrac{1}{2}cis\left(-\dfrac{\pi}{4}\right)$
Specific behaviours
✓ determines the correct modulus r ✓ determines the correct argument θ

The complex number z is represented in the Argand diagram below.



(b) Express z exactly in the form $z = a + bi$. (2 marks)

Solution
From the Argand diagram $z = 4cis\left(\dfrac{2\pi}{3}\right)$ Hence $z = 4\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right) = 4\left(-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right) = -2 + 2\sqrt{3}\,i$
Specific behaviours
✓ determines the polar form for z correctly (interprets the Argand diagram) ✓ determines the correct exact values for a, b

Question 12 (continued)

(c) Determine the exact polar form for wz and w^2z. (2 marks)

Solution
Given $w = \frac{1}{2}cis\left(-\frac{\pi}{4}\right)$ and $z = 4cis\left(\frac{2\pi}{3}\right)$ Then $wz = \frac{1}{2}\times 4cis\left(-\frac{\pi}{4}+\frac{2\pi}{3}\right) = 2cis\left(\frac{5\pi}{12}\right)$ Also $w^2z = \left(\frac{1}{2}\right)^2 \times 4\times cis\left(-\frac{\pi}{2}+\frac{2\pi}{3}\right) = 1cis\left(\frac{\pi}{6}\right)$
Specific behaviours
✓ determines the correct modulus for both wz and w^2z ✓ determines the correct argument for both wz and w^2z

(d) On the Argand diagram on page 6, plot the position for wz and w^2z. Ensure that each position is labelled clearly. (2 marks)

Solution
Indicated on the Argand diagram.
Specific behaviours
✓ indicates the correct modulus for both wz and w^2z (distance from origin) ✓ indicates the correct argument for both wz and w^2z (angle to real axis)

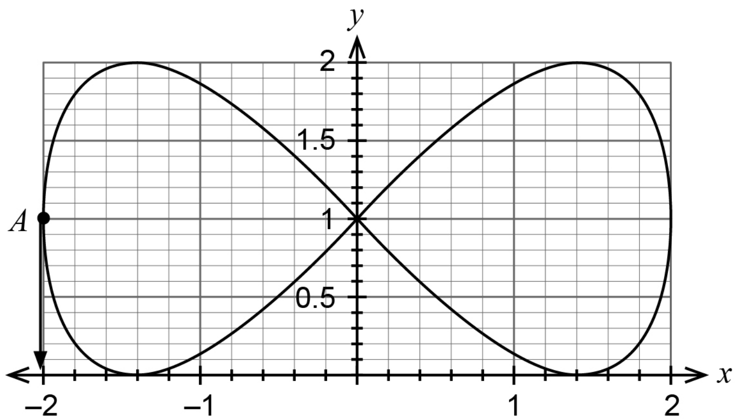
Consider the geometric transformation(s) applied to transform $z \to wz \to w^2z \to w^3z$ etc.

(e) Describe the geometric transformation(s) performed by the successive multiplication by w. (2 marks)

Solution
Successive multilplication by w results in the modulus changing by a factor of $\frac{1}{2}$ (successive points becoming twice as close to the origin) and the argument decreasing by 45° or $\frac{\pi}{4}$. Geometric description: Each vector is REDUCED by a factor of 0.5. Each vector is ROTATED clockwise (about origin) by 45°
Specific behaviours
✓ describes the change in the modulus a dilation by factor 0.5 ✓ describes the change in the argument as a clockwise rotation by 45° or $\frac{\pi}{4}$

Question 13 **(10 marks)**

The path of a particle is shown below. This particle moves so that its position vector $\underset{\sim}{r}(t)$ is given by $\underset{\sim}{r}(t) = \begin{pmatrix} -2\cos\left(\dfrac{t}{2}\right) \\ 1-\sin(t) \end{pmatrix}$ metres, where t is the number of seconds the particle has been in motion.



(a) Determine the starting position of the particle and mark this as point A on the diagram above. (1 mark)

Solution
Substituting $t=0$ $\underset{\sim}{r}(0) = \begin{pmatrix} -2\cos(0) \\ 1-\sin(0) \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
Specific behaviours
✓ indicates the point $(-2,1)$ on the diagram correctly

(b) Determine the initial velocity of the particle and illustrate this on the diagram above. (3 marks)

Solution
$\underset{\sim}{v}(t) = \dfrac{d\underset{\sim}{r}}{dt} = \begin{pmatrix} \sin\left(\dfrac{t}{2}\right) \\ -\cos(t) \end{pmatrix}$ Substituting $t=0$ $\underset{\sim}{v}(0) = \begin{pmatrix} \sin\left(\dfrac{0}{2}\right) \\ -\cos(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
Specific behaviours
✓ determines $\underset{\sim}{v}(t)$ by differentiating BOTH components correctly
✓ evaluates $\underset{\sim}{v}(0)$ correctly
✓ indicates a vector representing $\underset{\sim}{v}(0)$ on the diagram correctly

Question 13 (continued)

(c) Write the expression, in terms of trigonometric functions, for the distance the particle would travel in completing one circuit of the given path. Do **not** evaluate this expression. (3 marks)

Solution
One circuit is completed when $-2\cos\frac{t}{2} = -2$ i.e. $t = 0,\ 4\pi,\ \ldots$ Distance travelled for one circuit $= \int_0^{4\pi} Speed(t)\, dt \quad = \int_0^{4\pi} \lvert \underset{\sim}{v}(t) \rvert\, dt$ $= \int_0^{4\pi} \sqrt{\sin^2\left(\frac{t}{2}\right) + \cos^2(t)}\, dt$
Specific behaviours
✓ forms a definite integral using the correct limits $t = 0,\ 4\pi$ ✓ writes the correct expression for the speed function in terms of t using trigonometric functions   ✓ uses correct mathematics notation for vectors and the definite integral

(d) Determine the Cartesian equation for the path of the particle. (3 marks)

Solution
We have $x = -2\cos\left(\frac{t}{2}\right)$ i.e. $\cos\left(\frac{t}{2}\right) = -\frac{x}{2}$ and $y = 1 - \sin(t)$ i.e. $\sin(t) = 1 - y$ $\cos\left(2\left(\frac{t}{2}\right)\right) = 2\cos^2\left(\frac{t}{2}\right) - 1 \quad \therefore \quad \cos(t) = 2\left(-\frac{x}{2}\right)^2 - 1 = \frac{x^2}{2} - 1$ Since $\sin^2\theta + \cos^2\theta = 1$ then the Cartesian equation becomes $(1-y)^2 + \left(\frac{x^2}{2} - 1\right)^2 = 1$ Alternative forms: $1 - 2y + y^2 + \frac{x^4}{4} - x^2 = 0$ $(y-1)^2 = 1 - \left(\frac{x^2}{2} - 1\right)^2$ or $y = 1 \pm \sqrt{1 - \left(\frac{x^2}{2} - 1\right)^2}$
Specific behaviours
✓ forms correct expressions for $\cos\left(\frac{t}{2}\right)$ and $\sin(t)$ correctly in terms of x, y ✓ uses the double angle identity to obtain trigonometric equations with the same variable (either both in terms of $\frac{t}{2}$ or t)   ✓ uses the identity $\sin^2\theta + \cos^2\theta = 1$ correctly to eliminate t to obtain the Cartesian equation

Question 14 **(10 marks)**

Trucks carrying iron ore for the Croc Rock mining company arrive at a weighing station. The service time T per truck is defined to be the time elapsed from the moment a truck enters the station zone, including the time to be positioned and then weighed, up to the time it leaves the zone.

It is known that the population mean $\mu(T) =$ 80 seconds and the population standard deviation $\sigma(T) =$ 20 seconds.

At the Croc Rock weighing station, 100 trucks are weighed.

(a) State the (approximate) distribution of the sample mean service time per truck for the 100 trucks. (3 marks)

Solution
$\overline{T}$ is approximately normally distributed as the sample size $n = 100 > 30$ $\overline{T} \sim N\left(80, \dfrac{20^2}{100}\right) = N(80,\ 4) \qquad i.e.\ \sigma\left(\overline{T}\right) = \sqrt{4} = 2$
Specific behaviours
✓ states the sample mean is normally distributed (or refers to the Central Limit Theorem) ✓ states the correct mean ✓ states the correct standard deviation (or variance)

(b) What is the probability that the sample mean service time will be more than 83 seconds? (2 marks)

Solution
$P\left(\overline{T} > 83\right) = P\left(z > \dfrac{83-80}{2}\right) = P(z > 1.5)$ $= 0.067$
Specific behaviours
✓ calculates/uses the correct z scores ✓ determines the correct probability

Question 14 (continued)

Suppose that more than 100 trucks were weighed at the Croc Rock weighing station.

(c) How would this affect your answer to part (b)? Explain without recalculation. (2 marks)

Solution
Since for $n > 100$ would result in $\sigma\left(\overline{T}\right) < 2$, then the mean time of 83 minutes would be a more of an extreme sample mean in the normal distribution. Hence there would be a lower probability that $P\left(\overline{T} > 83\right)$. i.e. the answer to part (b) would be LOWER.
Specific behaviours
✓ States that the answer would decrease ✓ Justifies by reference to the lower standard deviation of the sample mean OR that 83 minutes becomes a relatively more extreme score

It is desired that the probability that the sample mean service time will be between 80 seconds and 82 seconds is greater than 40%.

(d) Determine the minimum number of trucks that will need to be weighed. (3 marks)

Solution
Require $P\left(80 < \overline{T} < 82\right) \ > \ 0.4$ i.e. $P\left(0 < z < k\right) \ > \ 0.4 \quad \therefore \quad k > 1.282$ Hence $\dfrac{82-80}{\dfrac{20}{\sqrt{n}}} \ > \ 1.282$ Solving gives $n > 164.35$ $\therefore$ At least 165 trucks are required to be weighed.
Specific behaviours
✓ calculates the critical z score for 82 minutes ✓ forms the inequality or equation to solve for the number of trucks ✓ states the minimum number of trucks to be weighed as an integer

Question 15 **(9 marks)**

A random sample of n commuters in Melbourne in August 2018 found that the average time to commute to work was 40 minutes. Repeated sampling of the mean indicated that the standard deviation of the sample mean was 3 minutes.

(a) Determine a 90% confidence interval for the population mean commuting time μ to work, correct to 0.01 minutes. (3 marks)

Solution
90% confidence interval for μ : $P(-k < z < k) = 0.9$ yields $k = 1.645$ $40 - 1.645(3) < \mu < 40 + 1.645(3)$ i.e. $35.06 < \mu < 44.94$ minutes
Specific behaviours
✓ uses the critical z score for 90% confidence ✓ forms the correct expression for the confidence interval limits (using $\sigma(\overline{X}) = 3$) ✓ determines the upper and lower limit correctly (no penalty for incorrect rounding)

Another random sample of $2n$ commuters in November 2018 found that the average time to commute to work was 45 minutes. Assume that both the August and November samples were drawn from the same population.

(b) What is the standard deviation of the sample mean for the November sample, correct to 0.01 minutes? (2 marks)

Solution
Let $\sigma =$ the population standard deviation for the commuting time (minutes) From August sample we have $\frac{\sigma}{\sqrt{n}} = 3$ i.e. $\sigma = 3\sqrt{n}$ For November sample we require : $\sigma(\overline{X}) = \frac{\sigma}{\sqrt{2n}} = \frac{1}{\sqrt{2}}\left(\frac{\sigma}{\sqrt{n}}\right)$ $= \frac{1}{\sqrt{2}}(3) = 2.12$ min (2 d.p.)
Specific behaviours
✓ forms the correct relationship between the standard deviations OR states that $\sigma = 3\sqrt{n}$ (or its equivalent) ✓ determines the standard deviation of sample mean correct to 0.01 minutes

Question 15 (continued)

Suppose that the August and November samples are combined to form a sample with $3n$ commuters. Consider 90% confidence intervals for the following samples for the purpose of determining the population mean commuting time μ.

90% confidence interval	Sample	Size
A	August	n
N	November	$2n$
C	Combined	$3n$

(c) Which of the three confidence intervals, A, N or C, will provide the greatest precision in determining the population mean μ? Justify your answer. (2 marks)

Solution
C is the most precise, as it is based on the smallest standard error (standard deviation of the sample mean)
Specific behaviours
✓ states that C will provide the greatest precision ✓ provides a valid reason (smallest standard deviation in the sample mean or largest sample size)

(d) Which of the three confidence intervals, A, N or C, contains the true value of the population mean μ? Justify your answer. (2 marks)

Solution
We do not know which interval contains the population mean μ. This is because we do not know the true value of μ OR that by random sampling we cannot be certain which interval contains μ.
Specific behaviours
✓ states that we do not know which interval contains the population mean ✓ provides a valid reason (either μ is not known OR that by random sampling we cannot be certain which interval contains μ)

Question 16 **(12 marks)**

Plane Π_1 has Cartesian equation $z = 2x + y + 4$.

(a) Determine a vector that is normal to plane Π_1. (2 marks)

Solution
Cartesian equation can be written as $2x + y - z = -4$.
$\therefore \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -4$ is the vector normal form for plane Π_1.
Hence the vector $2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ is perpendicular (the normal) to plane Π_1.
Specific behaviours
✓ re-writes the Cartesian form into vector form OR into standard Cartesian form.
✓ states the vector perpendicular to plane Π_1

Line L has equation $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(b) Determine the point of intersection between line L and plane Π_1. (3 marks)

Solution
Line L has equation $\underset{\sim}{r} = \begin{pmatrix} 2+\lambda \\ 2\lambda \\ 3-\lambda \end{pmatrix}$
Solving simultaneously: $\therefore \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2+\lambda \\ 2\lambda \\ 3-\lambda \end{pmatrix} = -4$
i.e. $2(2+\lambda) + 1(2\lambda) - 1(3-\lambda) = -4$
i.e. $5\lambda + 1 = -4$
i.e. $\lambda = -1$ Hence intersection point is $(1, -2, 4)$.
Specific behaviours
✓ substitutes the expression for line vector correctly into equation for plane
✓ solves correctly for the value of parameter λ
✓ states the point of intersection (can be either in vector or Cartesian form)

Question 16 (continued)

Plane Π_2 contains line L and is perpendicular to plane Π_1.

(c) Determine the vector equation for plane Π_2. (4 marks)

Solution
Let $\underset{\sim}{n}_2$ be the normal vector for plane Π_2. Hence $\underset{\sim}{n}_2$ is perpendicular to $\underset{\sim}{d}$ direction vector of the line L AND to the normal $\underset{\sim}{n}_1$ of plane Π_1. Hence $\underset{\sim}{n}_2 = \underset{\sim}{n}_1 \times \underset{\sim}{d} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ A point in plane Π_2 can be any point on Line L i.e. $(2,0,3)$ Vector equation for Π_2: $\underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 11$ Note: Cartesian equation for Π_2: $x + y + 3z = 11$
Specific behaviours
✓ states that the normal $\underset{\sim}{n}_2$ is perpendicular to both $\underset{\sim}{d}$ and $\underset{\sim}{n}_1$ ✓ determines the vector $\underset{\sim}{n}_2$ (by cross product or otherwise) ✓ uses correctly a point of L as a point for plane Π_2 ✓ forms the vector equation for plane Π_2 correctly

or

Alternative Solution
The vector equation for plane Π_2 is given by : $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ i.e. $\underset{\sim}{r} = \begin{pmatrix} 2+\lambda+2\mu \\ 2\lambda+\mu \\ 3-\lambda-\mu \end{pmatrix}$ This is because Π_2 contains line L so it contains the point $(2,0,3)$ and contains the direction vector $(1,2,-1)$ of line L. Also plane Π_2 has to be in a direction that is normal to Π_1 hence we can move in the direction $(2,1,-1)$.
Specific behaviours
✓ uses the known point $(2,0,3)$ in writing the vector equation using two directions ✓ uses two different parameters with the two directions ✓ uses the direction vector $(1,2,-1)$ for line L correctly ✓ uses the normal vector $(2,1,-1)$ for plane Π_1 correctly

Sphere S has vector equation $\left| \underset{\sim}{r} - \left(3\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}\right) \right| = \sqrt{35}$.

(d) Determine whether line L is a tangent to sphere S. Justify your answer. (3 marks)

Solution
Examine how many points of intersection there are between the line and the sphere. To find the intersection solve simultaneously : $\left\| \begin{pmatrix} 2+\lambda \\ 2\lambda \\ 3-\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\| = \sqrt{35}$ *i.e.* $(\lambda-1)^2 + (2\lambda-1)^2 + (-1-\lambda)^2 = 35$ $\therefore\ 6\lambda^2 - 4\lambda - 32 = 0$ $\therefore\ \lambda = -2$ or $\lambda = \frac{8}{3}$ $\therefore$ There are 2 points of intersection. $\therefore$ The line is NOT a tangent.
Specific behaviours
✓ obtains the equation that determines the intersection ✓ solves correctly to determine the parameter λ ✓ justifies that the 2 points of intersection means the line is not a tangent

or

Alternative Solution
Determine the shortest distance from the centre of the sphere to the line. Using point $A\ (2,0,3)$ on line L and centre of sphere $C\ (3,1,4)$. Hence $\overrightarrow{AC} = (1,1,1)$. Shortest distance from point C to line L $= \frac{\left\|\overrightarrow{AC} \times \underset{\sim}{d}\right\|}{\left\|\underset{\sim}{d}\right\|} = \frac{\|(1,1,1)\times(1,2,-1)\|}{\|(1,-2,1)\|} = \frac{\|(-3,2,1)\|}{\sqrt{6}} = \frac{\sqrt{14}}{\sqrt{6}} = \sqrt{\frac{7}{3}} = 1.527...$ Since the radius of sphere $\sqrt{35} > \sqrt{\frac{7}{3}}$ then there will be two points of intersection.
Specific behaviours
✓ forms the correct expression for the shortest distance ✓ calculates the shortest distance correctly ✓ justifies that the radius > shortest distance means the line is not a tangent

Question 17 **(8 marks)**

In Australia, the killing of humpback whales was banned in 1963.

At the end of 2018, 45 years later, the population P of migrating humpback whales off the coast of Western Australia was estimated at 30 000, i.e. $P(45) = 30\ 000$.

(a) Assuming that the population of humpback whales had been increasing at an instantaneous rate equal to 10% of the population, estimate the number of humpback whales at the end of 1963. (3 marks)

Solution using $P(45) = 30\ 000$
Rate of change is given by $\dfrac{dP}{dt} = 0.1P$. This has solution $P(t) \ = \ P_0 e^{0.1t}$ Using $P(45) = 30\ 000 \quad 30\ 000 \ = \ P_0 e^{0.1(45)}$ Solving gives $P_0 \ = \ 333.26...$ i.e. there were 333 humpback whales at the end of 1963. (answers 300 or 330 are acceptable given the initial information)
Specific behaviours
✓ forms the differential equation correctly to represent the rate of change ✓ writes the specific exponential solution for this differential equation ✓ uses $P(45) = 30\ 000$ to correctly deduce $P(0)$ as an integer value

or

Solution using $P(55) = 30\ 000$
Rate of change is given by $\dfrac{dP}{dt} = 0.1P$. This has solution $P(t) \ = \ P_0 e^{0.1t}$ Using $P(55) = 30\ 000 \quad 30\ 000 \ = \ P_0 e^{0.1(55)}$ Solving gives $P_0 \ = \ 122.60...$ i.e. there were 123 humpback whales at the end of 1963. (answers 100 or 120 are acceptable given the initial information)
Specific behaviours
✓ forms the differential equation correctly to represent the rate of change ✓ writes the specific exponential solution for this differential equation ✓ uses $P(55) = 30\ 000$ to correctly deduce $P(0)$ as an integer value

To model the growth in the population from the end of 2018, a marine biologist suggests that the rate of growth be modelled by the equation below.

$$\frac{dP}{dt} = 0.1P - \frac{P^2}{700\ 000}$$

The biologist re-defines $P(0) = 30\ 000$, i.e. $t =$ number of years from the end of 2018.

(b) If $P(t)$ is written in the form $P(t) = \dfrac{a}{1 + be^{-ct}}$, determine the values of the constants a, b and c. (2 marks)

Solution
$\frac{dP}{dt} = 0.1P - \frac{P^2}{700000} = \frac{P}{10} - \frac{P^2}{700000} = \frac{1}{700000}P(70000-P)$
From formula sheet $\frac{dP}{dt} = rP(k-P)$ gives $P = \frac{kP_0}{P_0 + (k-P_0)e^{-rkt}}$
As $P(0) = 30\ 000$, then $P(t) = \frac{(70000)(30000)}{30000 + (70000-30000)e^{-0.1t}}$
$\therefore\ P(t) = \frac{70000}{1 + \frac{4}{3}e^{-0.1t}}$ i.e. $a = 70000,\ b = \frac{4}{3},\ c = 0.1$
Specific behaviours
✓ determines the correct value of constant a ✓ determines the correct value of the constants b, c

(c) Hence determine the year during which the population of humpback whales off the coast of Western Australia will reach double that estimated at the end of 2018. (2 marks)

Solution
Require when $P(t) = 60000$.
i.e. Solve $\therefore\ 60000 = \frac{70000}{1 + \frac{4}{3}e^{-0.1t}}$ $\therefore\ e^{-0.1t} = \frac{1}{8}$ *i.e.* $t = 10(\ln 8)$
Using CAS $t = 20.794....$ years Hence the population is estimated to double during year 2039.
Specific behaviours
✓ forms the correct equation to solve for t ✓ solves and concludes the calendar year for the population to double

Question 17 (continued)

(d) State the major difference in the variation in the population $P(t)$ using the model in part (b) compared with that in part (a). (1 mark)

Solution
The rate of growth in part (b) will increase from $P = 30\ 000$ to $P = 35\ 000$ and then the rate of growth decreases from $P = 35\ 000$ to approach an equilibrium or steady state population of $P = 70\ 000$ i.e. part (b) model represents limited growth. In part (a), the model will predict that growth will continue without any limit.
Specific behaviours
✓ states a valid property that distinguishes the growth in part (b) from part (a)

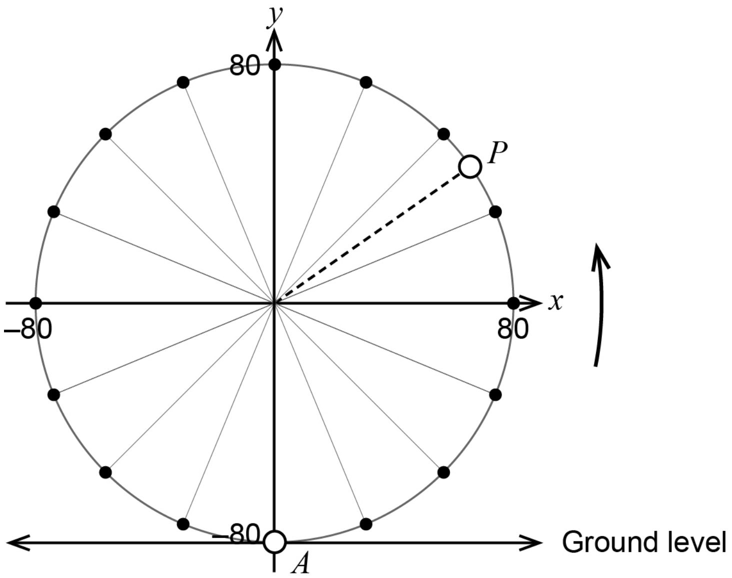
Question 18 **(11 marks)**

A ferris wheel has a radius of 80 metres and rotates in an anticlockwise direction at a rate of one revolution every 72 seconds. The ferris wheel has 16 cars that are equally spaced around the wheel as shown in the diagram.

A coordinate system is set up so that the centre of the ferris wheel is at the origin and the ground level has equation $y = -80$. Passengers begin their ride when a car is at position A $(0, -80)$.

Consider a passenger in a car at position P.

Let t = the number of seconds the ride has been in progress from position A.
θ = the angle in radians that the car has rotated from position A.
y = the height of a car above the centre of the ferris wheel (metres).



(a) Show that $\dfrac{d\theta}{dt} = \dfrac{\pi}{36}$ radians per second. (1 mark)

Solution
One revolution $(2\pi$ radians$)$ every 72 seconds means $\dfrac{d\theta}{dt} = \dfrac{2\pi}{72} = \dfrac{\pi}{36}$
Specific behaviours
✓ states that the angle changes 2π radians every 72 seconds

Question 18 (continued)

(b) Given that $y(\theta) = 80\sin(\theta+\alpha)$, explain why $\alpha = -\frac{\pi}{2}$. (1 mark)

Solution
Using $y(0)=-80$ then $-80=80\sin(\alpha)$ i.e. $\sin(\alpha)=-1 \;\therefore\; \alpha=-\frac{\pi}{2}$ OR at position P the angle above the x axis is equal to $\theta-\frac{\pi}{2} \;\therefore\; \alpha=-\frac{\pi}{2}$
Specific behaviours
✓ justifies correctly the value for α

(c) Determine how quickly a passenger is moving upward when they are 100 metres above the ground, correct to the nearest 0.01 metres per second. (4 marks)

Solution
When 100 metres above ground, $y=100-80=20$ $\therefore\; 20=80\sin\left(\theta-\frac{\pi}{2}\right)$ *i.e.* $\sin\left(\theta-\frac{\pi}{2}\right)=\frac{1}{4}$ i.e. $\theta=1.823...$ Hence $\cos\left(\theta-\frac{\pi}{2}\right) = \sqrt{1-0.25^2} = \frac{\sqrt{15}}{4} = 0.9682...$ OR $\therefore\; 20=80\sin\left(\frac{\pi t}{36}-\frac{\pi}{2}\right)$ *i.e.* $t=20.8955...$ sec   As $y=80\sin\left(\theta-\frac{\pi}{2}\right)$ then differentiating both sides with respect to t $\frac{dy}{dt} = \frac{dy}{d\theta}\times\frac{d\theta}{dt} = 80\cos\left(\theta-\frac{\pi}{2}\right)\times\frac{\pi}{36}$ $= 80\left(\frac{\sqrt{15}}{4}\right)\left(\frac{\pi}{36}\right) = 6.7596...$ m/sec   Hence the passenger is moving up at a rate of 6.76 m/sec (nearest 0.01)
Specific behaviours
✓ deduces $y=20$ ✓ differentiates correctly using the the chain rule to obtain an expression for $\frac{dy}{dt}$ ✓ substitutes either $\cos\left(\theta-\frac{\pi}{2}\right)=\frac{\sqrt{15}}{4}$ or $t=20.8955$ or $\theta=1.823...$ ✓ calculates the rate correct to 0.01 m/sec

(d) Show that function $y(t)$ satisfies the condition for simple harmonic motion. (2 marks)

Solution
Need to show that $y''(t) = -n^2 y(t)$
$y(t) = 80\sin\left(\frac{\pi t}{36} - \frac{\pi}{2}\right)$ Hence $y'(t) = 80\cos\left(\frac{\pi t}{36} - \frac{\pi}{2}\right) \times \frac{\pi}{36}$
$\therefore\ y''(t) = 80 \times \frac{\pi}{36} \times \left(-\sin\left(\frac{\pi t}{36} - \frac{\pi}{2}\right)\right) \times \frac{\pi}{36}$
$= -\left(\frac{\pi}{36}\right)^2 \times 80\sin\left(\frac{\pi t}{36} - \frac{\pi}{2}\right) = -\left(\frac{\pi}{36}\right)^2 y(t)$
Hence function $y(t)$ satisfies the definition for simple harmonic motion.
Specific behaviours
✓ differentiates twice correctly to find $y''(t)$ correctly
✓ shows that $y''(t) = -n^2 y(t)$ where it is evident that $n = \frac{\pi}{36}$

A different passenger happens to be in a car that is two cars ahead of a particular car on the ferris wheel.

(e) At what speed, correct to the nearest 0.01 metres per second, is the trailing passenger moving upward when the other passenger is moving downward at exactly the same speed? (3 marks)

Solution
Equal speeds will occur when the 2 cars are equidistant from the y axis.
Adjacent cars are separated by $\frac{2\pi}{16} = \frac{\pi}{8}$ radians
$\therefore$ For the trailing car $\theta = \frac{7\pi}{8}$ or $t = \frac{63}{2} = 31.5$ sec.
$\therefore\ Speed = y'\left(\frac{7\pi}{8}\right) = 80\cos\left(\frac{3\pi}{8}\right) \times \frac{\pi}{36}$
$= 2.6716....$ m/sec
Hence the equal speeds is 2.67 m/sec (nearest 0.01)
Specific behaviours
✓ states the position for each car when equal speeds are achieved
✓ determines the correct value for θ or t for equal speeds
✓ calculates the speed (no penalty for incorrect rounding)

Question 19 **(4 marks)**

Two parallel planes Π_1 and Π_2 have their equations given by:

$\Pi_1 \quad \underset{\sim}{r} \bullet \underset{\sim}{n} = 11$

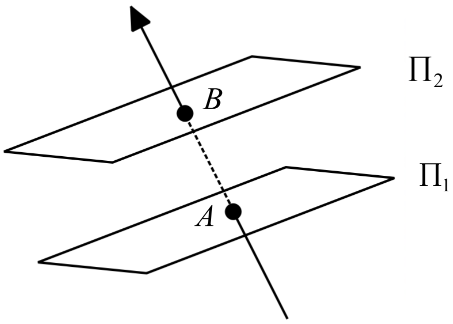
$\Pi_2 \quad \underset{\sim}{r} \bullet \underset{\sim}{n} = -4$ where $\underset{\sim}{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

It is known that the point (2,3,–7) is a point on plane Π_1.

Prove the distance d between the point (2,3,–7) and plane Π_2 is given by

$d = \dfrac{15}{\sqrt{a^2+b^2+c^2}}$.

Solution



Distance $d = AB$

Equation for line $\overrightarrow{AB}$ is given by:

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{(Equation 1)}$$

Since B is a point in plane Π_2 and line $\overrightarrow{AB}$ then $\left(\begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) \bullet \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -4$

i.e. $\begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix} \bullet \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -4$ (Equation 2)

i.e. $11 + \lambda\left(a^2+b^2+c^2\right) = -4$ $\quad \therefore \ \lambda = \dfrac{-15}{a^2+b^2+c^2}$ for position B.

$\lambda\left(a^2+b^2+c^2\right) = -4-11 = -15$

Hence distance $d = \left|\lambda \underset{\sim}{n}\right| = \left|\dfrac{-15}{a^2+b^2+c^2}\right| \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \dfrac{15}{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2}$

$= \dfrac{15}{\sqrt{a^2+b^2+c^2}}$

Specific behaviours

✓ forms the correct equation for the perpendicular to each plane
✓ substitutes correctly into the equation for plane Π_2 to form equation 2
✓ develops the correct expression for the parameter λ
✓ uses the idea that $d = \left|\lambda \underset{\sim}{n}\right|$ to determine the distance expression

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